C4/1 ALGEBRA

BINOMIAL EXPANSION OF $(1+x)^n$ FOR RATIONAL VALUES OF N

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

VALID $n \in \mathbb{Q}$ and $|x| \leq 1$

$$\frac{A}{C} \pm \frac{B}{D} = \frac{AD \pm BC}{CD}$$

$$\frac{A}{C} \times \frac{B}{D} = \frac{AB}{CD}$$

$$\frac{A}{C} \div \frac{B}{D} = \frac{AD}{CB}$$

$$\frac{x}{CD} = \frac{A}{C} + \frac{B}{D} \Rightarrow x = AD + BC$$

$d = 0 \Rightarrow B$

$C = 0 \Rightarrow A$

$$\frac{x}{C^2 D} = \frac{A}{C^2} + \frac{B}{C} + \frac{E}{D} \Rightarrow x = AD + BCD + EC^2$$

$D = 0 \Rightarrow E$

$C = 0 \Rightarrow A$

sub in A & E To get B.

LINEAR FACTORS $\frac{n}{(x-a)(x+b)} = \frac{A}{(x-a)} + \frac{B}{(x+b)}$

QUADRATIC FACTORS WHICH DONT HAVE FACTORS $\frac{n}{(x+k)(x^2-l)} = \frac{A}{(x+k)} + \frac{B}{(x^2-l)}$

REPEATED FACTORS $\frac{n}{(x+k)(x-p)^2} = \frac{A}{(x+k)} + \frac{B}{(x+p)} + \frac{C}{(x+p)^2}$

C4/2 TRIGONOMETRY

$\sin^2 A + \cos^2 A \equiv 1 \qquad 1 + \cot^2 A \equiv \operatorname{cosec}^2 A$

$\tan^2 A + 1 \equiv \sec^2 A$

$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad A+B \neq (k+\tfrac{1}{2})\pi$

$\sin 2A \equiv 2\sin A \cos A$

$\cos 2A \equiv 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$

$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha) / R\sin(\theta \pm \alpha)$

$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = a$

$R\sin\alpha = b \qquad \alpha = \tan^{-1}(b/a) \qquad R = \sqrt{a^2+b^2}$

$a\cos\theta \pm b\sin\theta = \sqrt{a^2+b^2}\cos(\theta \mp \tan^{-1}(b/a))$

$= \sqrt{a^2+b^2}\sin(\theta \pm \tan^{-1}(a/b))$

greatest & least values for $\cos\theta$ and $\sin\theta$ are 1 & -1

C4/3 CARTESIAN & PARAMETRIC EQUATIONS

CARTESIAN - CONNECT x AND y IN SOME WAY $\therefore y = f(x)$

PARAMETRIC - CONNECT x AND y IN TERMS OF T $\therefore y = f(t), x = g(t)$

PARAMETRIC → CARTESIAN : eliminate parameter 't' leaving equation relating y to x.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx} = \frac{d(dy/dx)}{dt} \times \frac{dt}{dx}$$

use $y - y_1 = m(x - x_1)$

IMPLICIT DIFFERENTIATION

$F'(x)$ as normal

$F'(y)$ with respect to y then $\times \frac{dy}{dx}$

solve for dy/dx

C4/4 INTEGRATION

INTEGRATION BY SUBSTITUTION

$\int F(x).g(x)\,\delta x$ let $u = f(x)$ $\quad F(u) = g(x)$

$\frac{du}{dx} = n \qquad dx = \frac{du}{n}$

$\Rightarrow \int u.f(u)\frac{du}{n}$ sub in $f(x) = u$

Trigonometric substitution can make integration easier

INTEGRATION BY PARTS

$$\int u.dv = uv - \int v.du$$

u = polynomial // u = other function
dv = easily integrated function // dv = polynomial

use partial fractions to simplify integration

VOLUME OF REVOLUTION

$\pi \int_a^b y^2\,dx$ The area under the curve rotated through four right angles (360°) forming a 3D shape.

FIRST ORDER DIFFERENTIAL

$\frac{dy}{dx} = xy$

$\frac{1}{y}\,dy = x\,\delta x$

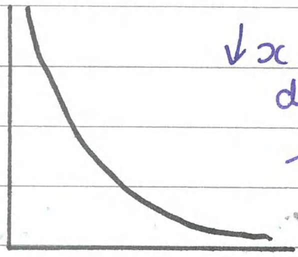
$\ln y = \frac{x^2}{2}$

$\ln y = \frac{x^2}{2} + c$

$y = e^{x^2/2 + c}$

$y = Ae^{x^2/2}$

C4/5 EXPONENTIAL DIFFERENTIALS

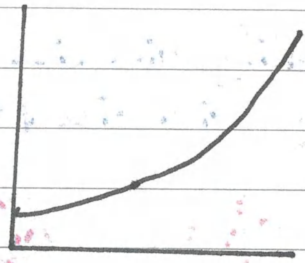


$\downarrow x \propto x$

$dx/dt \propto -x \quad \frac{dx}{dt} = -Kx$

$T = T_0 e^{-kt}$

DECAY



$\uparrow x \propto x$

$dx/dt = kx$

$T = T_0 e^{kt}$

GROWTH

$\int \frac{1}{x}\,dx = -k\int dt \Rightarrow \ln(x) = -kt + c$

when $t=0 \quad x = x_0$

$\ln(x_0) = 0 + c \Rightarrow \ln(x_0) = c$

$\ln(x) = \pm kt + \ln(x_0)$

$\ln(x/x_0) = \pm kt$

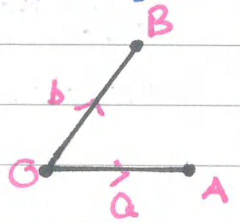
$x/x_0 = e^{\pm kt}$

$\frac{dP}{dt} \propto F(P)$

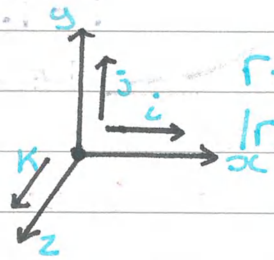
$\frac{dP}{dt} = kF(P) \quad \therefore \int \frac{1}{F(P)} = \int k\,dt$

C4/6 VECTORS

SCALAR : a quantity with size only
VECTOR : quantity with size & direction



$\overrightarrow{OA} = a$ $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$\overrightarrow{OB} = b$ $\quad \overrightarrow{AB} = b - a$



$r = ai + bj + ck$
$|r| = \sqrt{a^2 + b^2 + c^2}$

$a = x_1 i + y_1 j + z_1 k \qquad b = x_2 i + y_2 j + z_2 k$

$a + b = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$
$a - b = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$
$ra = r(x_1 i + y_1 j + z_1 k)$
$ra + sb = (rx_1 + sx_2)i + (ry_1 + sy_2)j + (rz_1 + sz_2)k$

$y = mx + c$
r direction vector λ position vector

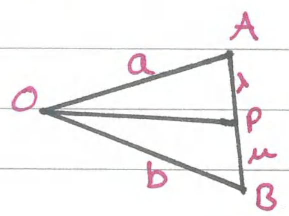
The equation of a straight line can be found if :
- Known direction and passes through a point
- has two known points.

Distance between two points $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$r = a + \lambda b$
$r = c + \mu d$ $\Rightarrow$ $a + \lambda b = c + \mu d$

Dot product $a.b = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\frac{a.b}{|a||b|} = \cos\theta$ $\quad \theta$ = angle between two vectors a and b
when perpendicular $a.b = 0$



$\overrightarrow{OP} = \frac{\mu a + \lambda b}{\lambda + \mu}$

FP1/1 MATHEMATICAL INDUCTION

$$\sum_{n=a}^{b} F(n)$$

a = n increases by
b = up to limit
F(n) = Function for each number
Σ = sum of

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$ sum of first 'n' natural numbers

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$ sum of first 'n' square numbers

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$ sum of first 'n' cubed numbers

METHOD OF DIFFERENCES

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} \qquad 1 = A(r+1) + B(r) \Rightarrow \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

$$\sum_{r=1}^{n} = \frac{1}{r} - \frac{1}{r+1}$$

$r=1$ $\frac{1}{1} - \frac{1}{2}$
$r=2$ $\frac{1}{2} - \frac{1}{3}$ $= 1 - \frac{1}{n+1}$
$r=3$ $\frac{1}{3} - \frac{1}{4}$ $= \frac{n}{n+1}$
$\vdots$
$r=n-1$ $\frac{1}{n-1} - \frac{1}{n}$ $\lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$
$r=n$ $\frac{1}{n} - \frac{1}{n+1}$

sum to ∞ tends to 1

PROOF BY INDUCTION

- Prove that proposition is true for trivial case: p(1) true
- Assume that p(k) true and use it to show p(k+1) true
- hence proved by induction for all positive integers

FP1/2 COMPLEX NUMBERS

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

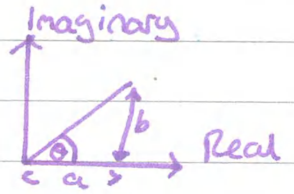
$i^4 = 1$

$i^5 = i$

complex number : $a + ib = z$

complex conjugate: $a - ib = \bar{z}$

$a \pm bi = c \pm di \Rightarrow a = c \Rightarrow b = d$



$[r, \theta]$ modulus $= r = \sqrt{a^2+b^2}$

argument $= \theta = \tan^{-1}(b/a)$ $\quad r\sin\theta = b$

$a + ib = r\cos\theta + ir\sin\theta$ $\quad r\cos\theta = a$

$z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ $\quad z_1z_2 = r_1r_2\cos(\theta_1+\theta_2) + ir_1r_2\sin(\theta_1+\theta_2)$

$z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$

$(a \pm bi) + (c + di) = (a+c) + (b+d)i$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\quad |z_1z_2| = |z_1|.|z_2|$ $\quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ $\quad \arg(z_1z_2) = \arg z_1 + \arg z_2$

$\arg z^n = n\arg z$ $\quad a + bi + x + iy = c + di$

$\frac{c \pm di}{a+bi} \times \frac{a-bi}{a-bi}$ $\quad a + x = c$

$b + y = d$

SOLVING EQTNS: complex roots are in conjugate pairs

- provided coefficients are real the equations for roots α, β, γ apply
- If $x - a$ is a root of $f(x) = 0$ $\quad f(a) = 0$

$|z - z_1| = r$ $\quad$ circle centre (a, b) radius r

$(x-a)^2 + (y-b)^2 = r^2$ $\quad z = x + iy$ $\quad z_1 = a + ib$

$|z - z_1| = |z - z_2|$ $\quad$ perpendicular bisector of lines

$(y-b)^2 + (y-d)^2 = (x-c)^2 - (x-a)^2$ $\quad z_1 \,\&\, z_2$

$|z - z_1| = K|z - z_2|$

$(x-a)^2 + (y-b)^2 = K^2(x-c)^2 + K(x-d)^2$

circle, at any point on its radius is K times further from z_2 than z_1

TRANSFORMATIONS $\quad w = f(z)$

$z = f(w)$

$|z| = |f(w)|$

FP1/3 SUM & ROOTS OF POLYNOMIALS

FROM THE EQUATION

$$ax^2 + bx + c = 0$$

WITH ROOTS

$$\alpha \text{ and } \beta$$

$$(x-\alpha)(x-\beta) = 0$$

$$\therefore \quad x^2 - x(\alpha+\beta) + \alpha\beta = 0$$

comparing this with $ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

we can see that: $\boxed{\alpha+\beta = -b/a}$ and $\boxed{\alpha\beta = c/a}$

FROM THE EQUATION

$$ax^3 + bx^2 + cx + d = 0$$

WITH ROOTS

$$\alpha, \beta \text{ and } \gamma$$

$$\therefore \quad (x-\alpha)(x-\beta)(x-\gamma) = 0$$

comparing coefficients:

$$x^2 \Rightarrow -\alpha x^2 - \beta x^2 - \alpha x^2 = b/a$$

$$\boxed{\alpha+\beta+\gamma = -b/a}$$

$$x^1 \Rightarrow x(-\beta)(-\gamma) + x(-\alpha)(-\beta) + x(-\alpha)(-\gamma) = c/a$$

$$\boxed{\alpha\beta + \beta\gamma + \alpha\gamma = c/a}$$

$$x^0 \Rightarrow (-\alpha)(-\beta)(-\gamma) = d/a$$

$$\boxed{\alpha\beta\gamma = -d/a}$$

FP1/4 MATRICES

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $r \times c$ matrix

MANIPULATING

you can only multiply: $(m \times n).(n \times k) = (m \times k)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ $AB \neq BA$

IDENTITY MATRIX - multiplys with n to give n $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- a matrix multiplied by its inverse gives the identity matrix $A.A^{-1} = I$

$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ where $\frac{1}{ad-bc}$ = determinant

INV OF: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ => determinant $= a(ei-fh) - b(di-fg) + c(dh-eg) = K$

=> matrix of minors $\begin{pmatrix} m & n & o \\ p & q & r \\ s & t & u \end{pmatrix}$

$m = ei-fh$ $o = dh-eg$ $q = ai-cg$ $s = bf-ce$

$n = di-fg$ $p = bi-ch$ $r = ah-bg$ $t = af-cd$

$u = ae-bd$

=> apply $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to matrix of minors $\rightarrow \begin{pmatrix} m & -n & o \\ -p & q & -r \\ s & -t & u \end{pmatrix}$

=> transpose $\left(\diagdown\!\!\!\!\diagup\right)$: $\begin{pmatrix} m & -p & s \\ -n & q & -t \\ o & -r & u \end{pmatrix}$ (adjugate) $\rightarrow$ multiply by determinant $M^{-1} = \frac{1}{K}\begin{pmatrix} m & -p & s \\ -n & q & -t \\ o & -r & u \end{pmatrix}$

$dx + ey = f$

$ax + by = c$ $\underbrace{\begin{pmatrix} a & b \\ d & e \end{pmatrix}}_{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$ $I^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = I^{-1}\begin{pmatrix} c \\ f \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x + 2y = 9$
$4x + 5y = 5$ => $\left(\begin{array}{cc|c} 3 & 2 & 9 \\ 4 & 5 & 5 \end{array}\right)$ ← ECHELON FORM

you need the form: $\left(\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right)$ to solve

① unique solution - consistent & independent $\left(\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right)$

② infinitely many solutions - consistent & dependent $\left(\begin{array}{cc|c} 1 & a & b \\ 0 & 0 & 0 \end{array}\right)$

③ no solutions - inconsistent $\left(\begin{array}{cc|c} 1 & a & b \\ 0 & 0 & c \end{array}\right)$

① unique solution - consistent & $\left(\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right)$

② infinitely many solutions - consistent $\left(\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 0 & 0 \end{array}\right)$

③ no solutions - inconsistent $\left(\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 0 & f \end{array}\right)$

FP1/5 TRANSFORMATIONS

ISOMETRY - ALL LENGTHS REMAIN THE SAME
↳ rotation, reflection, translation

- ANTICLOCKWISE ROTATION THROUGH θ ABOUT O

$$r_\theta : \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reflection) in $y = \tan\theta x$

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

REFLECTION MATRIX

- reflection in the line through O at angle θ to x axis

$$q_\theta : \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TRANSLATION MATRIX

write translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$ as $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}$$

INVARIANT POINT - a point that is mapped onto itself

INVARIANT LINE - a line all of whose image points are mapped onto the line

IF $x' = f(x)$, $y' = f(y)$ then set $x' = x$, $y' = y$ and solve to find invariant points

FP1/6 DIFFERENTIATION

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

eg $\quad y = \dfrac{x}{x+2}$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left\{ \frac{x+h}{x+h+2} - \frac{x}{x+2} \right\}$$

$$= \frac{1}{h}\left\{ \frac{(x+h)(x+2) - x(x+h+2)}{(x+h+2)(x+2)} \right\}$$

$$= \frac{1}{h}\left\{ \frac{\cancel{x^2} \cancel{+2x} \cancel{+hx} + 2h \cancel{-x^2} \cancel{-hx} \cancel{-2x}}{(x+h+2)(x+2)} \right\}$$

$$= \frac{1}{\cancel{h}}\left\{ \frac{2\cancel{h}}{(x+h+2)(x+2)} \right\}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2}{(x+h+2)(x+2)} = \frac{2}{(x+2)^2}$$

LOGARITHMIC DIFFERENTIATION

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

FP2/1 FURTHER PARTIAL FRACTIONS

Three types of Partial Fractions

- LINEAR Factors in the denominator.

$$\frac{n}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

- REPEATED LINEAR Factors in the denominator

$$\frac{n}{(x+a)(x+b)^2} = \frac{A}{x+a} + \frac{B}{x+B} + \frac{C}{(x+b)^2}$$

- QUADRATIC Factor in the denominator

$$\frac{n}{(x^2+a)(x+b)} = \frac{Ax+B}{x^2+a} + \frac{C}{x+b}$$

Improper Fractions are where the power of the numberator is equal to or greater than the power of the denominator

$$\frac{x^n}{x^a} \quad a \geq n$$

$\frac{ax+b}{x+c}$ must be ~~w~~rewritten as $a + \frac{d}{x+c}$

FP2/2 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof:

p(1) true : $(\cos\theta + i\sin\theta)^1 = \cos(1\theta) + i\sin(1\theta)$

assume p(k) true: $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

consider p(k+1): $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k . (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + -\sin k\theta\sin\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + (i\sin\theta\cos k\theta + i\sin k\theta\cos\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

thus p(k+1) true => true for all positive integers.

$(a\cos\theta + ci\sin\alpha)(b\cos\beta + di\sin\gamma)$

$= |a|\times|b|\cos(\theta+\beta) + i|c|\times|d|\sin(\alpha+\gamma)$

$z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) - i\sin(n\theta)$

$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= i2\sin n\theta$

n^{th} root of a complex eg ; $z^3 = 1$ modulus $= 1 = r$

$\cos 0 + i\sin 0 = 1$ argument $= 0 = \theta$

$\cos\theta$ and $\sin\theta$ are periodic at 2π

$z = [\cos(2k\pi) + i\sin(2k\pi)]^{1/3}$ $k=0$ $\cos 0 + i\sin 0 = 1$

$= \cos(\frac{2}{3}k\pi) + i\sin(\frac{2}{3}k\pi)$ $k=1$ $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z = 1, z = \frac{-1}{2} \pm \frac{\sqrt{3}\pi}{2}$ $k=2$ $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

FP2/3 TRIGONOMETRY

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

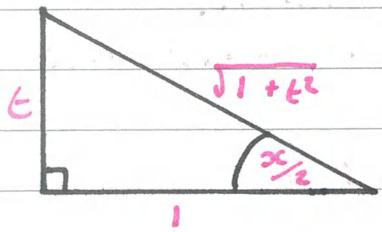
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

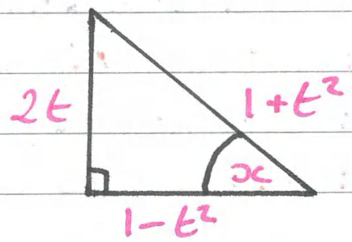
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\overset{\cos}{\cancel{\sin}}\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

t - SUBSTITUTION

$\left.\begin{array}{l}\sin A \pm \sin B = c \\ \cos A \pm \cos B = c\end{array}\right\}$ can be solved using $t = \tan\left(\frac{x}{2}\right)$





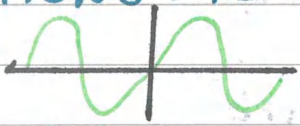
since $\tan x = \frac{2t}{1-t^2} \Rightarrow \sin x = \frac{2t}{1+t^2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$

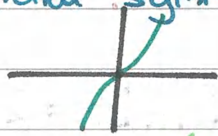
GENERAL SOLUTIONS FOR TRIGONOMETRIC EXPRESSIONS

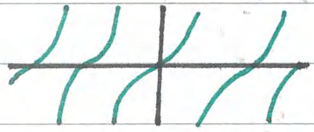
$\tan x = \alpha$	$\cos x = \alpha$	$\sin x = \alpha$
$\tan^{-1}\alpha + n\pi$	$2n\pi \pm \cos^{-1}\alpha$	$n\pi + (-1)^n \sin^{-1}\alpha$

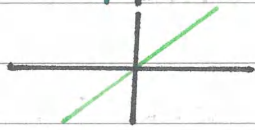
FP2/4 FUNCTIONS

ODD FUNCTIONS: $f(-x) = -f(x)$, rotational symmetry about O

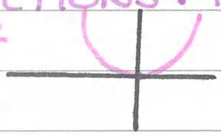
$f(x) = \sin x$ 

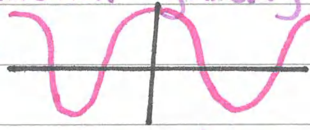
$f(x) = x^3$ 

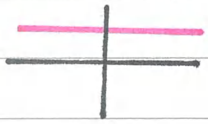
$f(x) = \tan x$ 

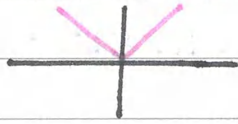
$f(x) = x$ 

EVEN FUNCTIONS: $f(-x) = f(x)$, reflectional symmetry in y axis

$f(x) = x^2$ 

$f(x) = \cos x$ 

$f(x) = a$ 

$f(x) = |x|$ 

MONOTONIC FUNCTIONS: strictly increasing/decreasing. no turning points.
-can have points of inflections.

BOUNDED FUNCTIONS: have a heighest and/or lowest point

CONTINUOUS FUNCTIONS: have no asymptotes

PERIODIC FUNCTIONS: repeats at a set time

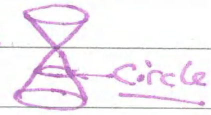
CURVE SKETCHING

- any obvious symmetry? - odd/even
- look where it crosses the x and y axis
- look at behaviour as $x \to \pm\infty$, $y \to \pm\infty$
- any undefined points - asymptotes
- turning points or points of inflection
- regions y cannot take - $b^2 - 4ac \geq 0$

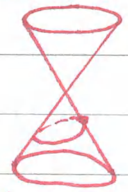
An oblique asymptote is a slanty asymptote which tells you what x tends to as it gets very large. Only oblique when the power on the top is larger than that on the bottom.

FP2/5 CONIC SECTIONS

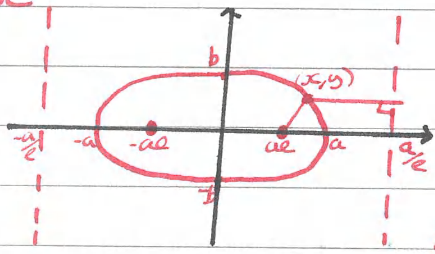
CONIC SECTIONS FOUND BY CUTTING A DOUBLE CONE AT DIFFERENT ANGLES e.g. CIRCLE - perpendicular to axis



ELLIPSE



cut at a slant



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

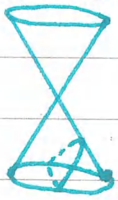
Parametric $(a\cos\theta, b\sin\theta)$

Foci : $(\pm ae, 0)$ Directrices $x = \pm \frac{a}{e}$

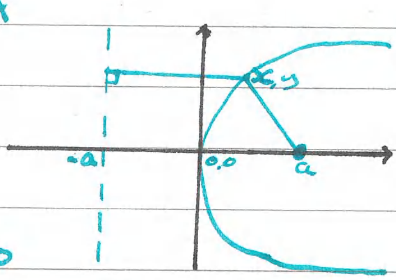
Eccentricity = e = ratio $ae \to x,y : \frac{a}{e} \to x,y$

$= e < 1$, $b^2 = a^2(1-e^2)$

PARABOLA



cut parallel to slant of side of cone



$y^2 = 4ax$

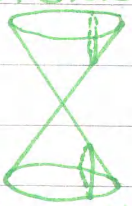
Parametric $(at^2, 2at)$

Foci : $(a, 0)$ Directrices $x = -a$

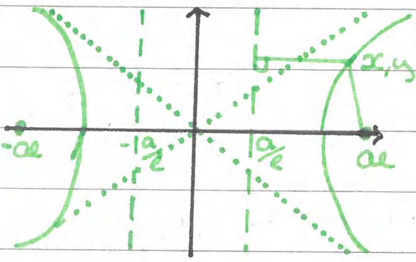
Eccentricity = e = ratio $a \to x,y : -a \to x,y$

$= e = 1$

HYPERBOLA



cut parallel to axis of cone



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parametric $(a\sec\theta, b\tan\theta)$

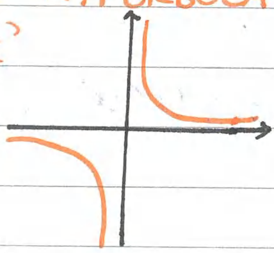
Foci : $(\pm ae, 0)$ Directrices $x = \pm a/e$

Eccentricity : $e > 1$ $b^2 = a^2(e^2 - 1)$

Asymptotes : $\frac{x}{a} = \pm \frac{y}{b}$

RECTANGULAR HYPERBOLA

where $a = b =$ 'c'



$xy = c^2$

Parametric $(ct, \frac{c}{t})$

Foci : $(\pm\sqrt{2}c, \pm\sqrt{2}c)$ Directrices $x + y = \pm\sqrt{2}c$

Eccentricity $e = \sqrt{2}$

Asymptotes $x = 0, y = 0$

FP2/6 FURTHER INTEGRATION

INTEGRATE $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$

$$\int \frac{1}{a^2+x^2}\,\delta x = \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\delta x = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad |x| < a$$

The coefficient <u>MUST</u> be one so take out a factor

Integrate using partial fractions and available variable limits

if: $I = \int_a^x F(t)\,dt$ then $\frac{dI}{dx} = Fx$

consider $I = \int_{u(x)}^{v(x)} f(t)\,dt$

$$= \left[F(t)\right]_{u(x)}^{v(x)} \qquad F'(t) = f(t)$$

$$I = F\{v(x)\} - F\{u(x)\}$$

$$\frac{dI}{dx} = F'\{v(x)\}v'(x) - F'\{u(x)\}u'(x)$$

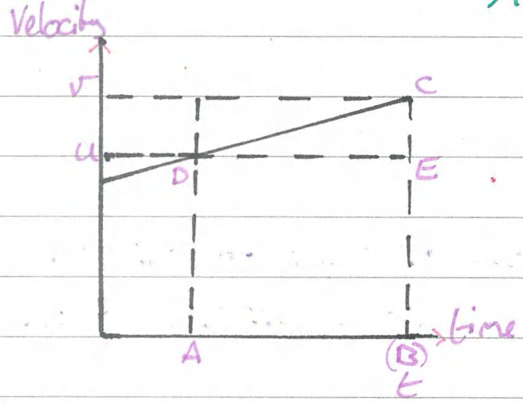
$$\frac{dI}{dx} = f\{v(x)\}\frac{dv}{dx} - f\{u(x)\}\frac{du}{dx}$$

$$I = \int_{u(x)}^{v(x)} f(t)\,dt$$

$$\frac{dI}{dt} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

M1/1 RECTILINEAR MOTION

$$ms^{-1} \xrightarrow{\times 18/5} Kmh^{-1}$$
$$ms^{-1} \xleftarrow[\times 5/18]{} Kmh^{-1}$$



Gradient : $a = \frac{v-u}{t}$

Neg gradient = deceleration
Pos gradient = acceleration

Area under curve : $s = \frac{1}{2}(u+v)t$

$a = \frac{v-u}{t}$

$at = v-u$

$[v = u+at]$

$s =$ Area ABDE + Area CDE

$s = ut + \frac{1}{2}(v-u)t$

$s = ut + \frac{1}{2}(at)t$

$[s = ut + \frac{1}{2}at^2]$

$v^2 = (u+at)(u+at)$

$v^2 = u^2 + 2aut + a^2t^2$

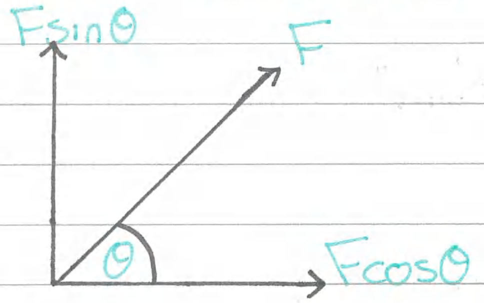
$v^2 = u^2 + 2a(ut + \frac{1}{2}at^2)$

$[v^2 = u^2 + 2as]$

$\left[s = \frac{(u+v)t}{2}\right]$

MODEL ASSUMPTIONS:
- body is considered a point mass
- air resistance is ignored
- motion of the body is (observed) along a vertical line
- the acceleration is due to gravity.

M1/2 Forces & Newtons Laws of Motions



Find component forces vertically & horizontally then find overall resultant force:

$F = \sqrt{(F\sin\theta)^2 + (F\cos\theta)^2}$

1ST LAW: every body continues in its state of rest or uniform motion in a straight line unless compelled by some external force.

2ND LAW: The force acting on an object equals the product of the objects mass and the acceleration produced. Force equals change in momentum per second → $F = m.a$

3rd LAW: ACTION AND REACTION are equal and opposite and act on different bodies

WEIGHT: attractive force between earth and particle in downwards direction

weight = mass × g

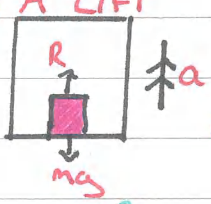
FRICTION: Force which opposes movement if surface is rough

NORMAL: the force a surface exerts on a particle

TENSION: resisting force in a string which opposes tendancy for the string to extend

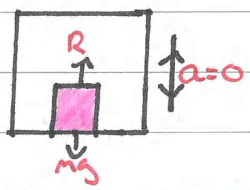
THRUST: resisting force provided by spring in opposing direction to force compressing/extended the string.

IN A LIFT



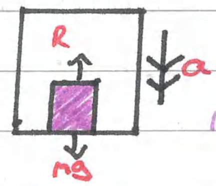
$R > mg$

$ma = R - mg$



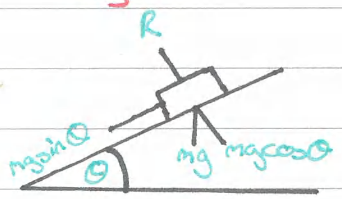
$R = mg$

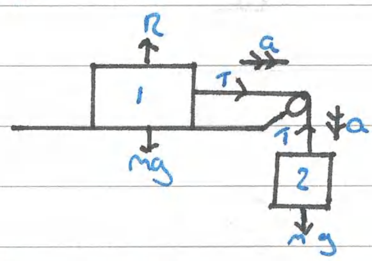
$ma = 0$



$R < mg$

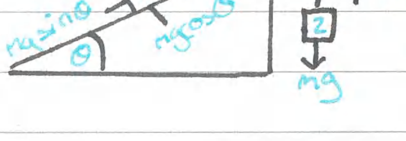
$ma = mg - R$





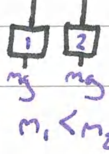
$m_1 a = T$

$m_2 a = m_2 g - T$



$m_1 a = T - m_1 g\sin\theta$

$m_2 a = m_2 g - T$

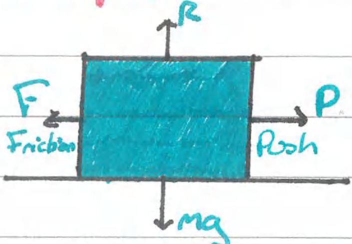


$m_1 < m_2$

$ma = T - m_1 g$

$ma = m_2 g - T$

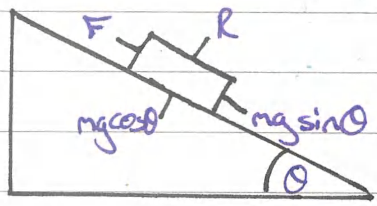
M1/3 FRICTION



IF SURFACE IS PERFECTLY SMOOTH $F = 0$

- when surfaces are rough, if $P > F$ the block will move
- The frictional force is not constant
- F increases as the applied force [P] increases
- until F reaches F_{max} where $F_{max} = \mu R$
- when $F = F_{max}$ the block is on the point of moving and is said to be on limiting equilibrium.
- μ is the coefficient of friction
- $\mu = 0$ perfectly smooth surface
- $\mu = 1$ maximum friction at any P
- $F \leq \mu R$ $0 \leq \mu \leq 1$

It's easier to move a block if pulling force is inclined upwards as this reduces frictional force opposing motion



$R = mg\cos\theta$

$F_{max} = mg\sin\theta$

$F_{max} = \mu R$

$mg\sin\theta = \mu mg\cos\theta$

$\mu = \tan\theta$

LAWS

- For two bodies in contact, the force of friction opposes the relative motion of the bodies
- If bodies are in equilibrium the force of friction is just sufficient to prevent motion
- The size of frictional force which can be exerted is limited
- Limited friction is the frictional force exerted when the equilibrium is on the point of being broken.

M1/4 MOMENTUM & IMPULSE

MOMENTUM OF AN OBJECT = MASS x VELOCITY

$= mv$

$= kgms^{-1}$ or Ns

THE IMPULSE OF AN OBJECT = FORCE x TIME

$= Ft$

= CHANGE IN MOMENTUM

CONSERVATION OF MOMENTUM

total momentum before = total momentum after

IMPACT OF BODIES ; Newtons Experimental Law

$$\text{coefficient of restitution } (e) = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$e = 0$ - ~~totally~~ inelastic impact

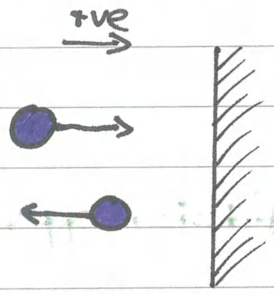
$e = 1$ - perfect elastic impact

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

IMPACT WITH A VERTICAL SURFACE

BEFORE

AFTER



$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{0 - v_1}{u_1 - 0} = \frac{-(-v)}{u} = \frac{v}{u}$$

M1/5 STATICS

the MOMENT of a force about a point, found by multiplying Magnitude by the perpendicular distance from the point to its line of action

ABOUT A POINT IN EQUILIBRIUM:

ANTICLOCKWISE MOMENTS = CLOCKWISE MOMENTS

CENTRE OF MASS (GRAVITY)

- the point at which the entire mass is assumed to be situated.
- If a lamina is suspended from any point then it will rest so that the line between the point of suspension and the CoM is vertical.

CENTRE OF MASS OF:

- RECTANGLE - where the lines joining the midpoints of the sides is situated
- CIRCLE : at centre of circle
- TRIANGLE : on the median $\frac{2}{3}$ from vertex

COMPOSITE SHAPES - LABEL EACH SHAPE

	Area	x	Ax	y	Ay
1.					
2.					

$A\bar{x} = \sum Ax$ $A\bar{y} = \sum Ay$

$(\bar{x}, \bar{y})$ = centre of mass

$\tan\theta = \bar{x}/\bar{y}$

When a series of particles is connected by light rods, the rods are considered to have no mass.

The mass of each particle is recorded along with the distance to a horizontal and vertical axis. Moments can then be taken about each axis

M2/1 RECTILINEAR MOTION

acceleration can be seen as a function of time

	DISPLACEMENT	VELOCITY	ACCELERATION
	x	v	a
DIFFERENTIATE		dx/dt	d^2x/dt^2 dv/dt
INTEGRATE	$\int(\int a\,dt)dt$ $\int v\,dt$	$\int a\,dt$	

cannot use SUVAT unless acceleration is constant

M2/2 DYNAMICS OF A PARTICLE

HOOKE'S LAW - The tension in a stretched string is proportional to the extension of the string from its natural length

$$T = \frac{\lambda x}{l}$$

T = tension x - extension

λ = modulus of elasticity l = natural length

Work done = FORCE x DISTANCE

Loss in Energy = work done against Force

Gain in Energy = work done by Force

Kinetic Energy = $\frac{1}{2}mv^2$

(Gravitational) Potential Energy = mgh

K.E. gained = P.E. lost

P.E. gained = K.E. lost

Total Energy = K.E. + P.E

ELASTIC POTENTIAL ENERGY

$T = \frac{\lambda x}{l}$ → work done by tension

$$\int_0^x \frac{\lambda x}{l}\,dx = \left[\frac{\lambda x^2}{2l}\right]_0^x \quad \therefore \text{EPE} = \frac{\lambda x^2}{2l}$$

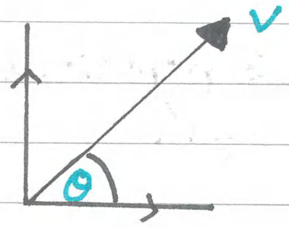
Power = Rate of work done

= Force x velocity

= watt

= Js^{-1}

M2/3 PROJECTILES

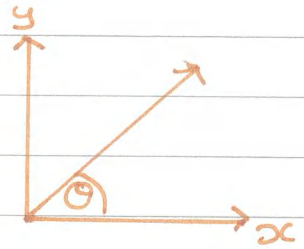


- horizontal velocity $= v\cos\theta$ we consider this to be constant throughout flight
- vertical velocity $= v\sin\theta$, subject to SUVAT

$v = u + at$
$v^2 = u^2 + 2as$
$s = ut + \frac{1}{2}at^2$
$s = (u+v)\frac{1}{2}t$

Given that the start/finish are at an equal level the time of flight is twice the time to get to the max height.

equation of trajectory:



$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

u = projectile initial velocity

GREATEST HEIGHT : $h = \dfrac{u^2\sin^2\theta}{2g}$

TIME OF FLIGHT : $T = \dfrac{2u\sin\theta}{g}$

HORIZONTAL RANGE : $R = \dfrac{u^2\sin 2\theta}{g}$

MAX HORIZONTAL RANGE : $R_{MAX} = \dfrac{u^2}{g}$

M2/4 VECTORS

- $a.b = |a||b|\cos\theta$ θ = angle between a and b
- $i.i = j.j = k.k = 1$ and $i.j = j.k = i.k = 0$
- $a.b = a_1b_1 + a_2b_2 + a_3b_3 + ... + a_nb_n$ in 'n' Dimensions
- $\cos\theta = \frac{a.b}{|a||b|}$ where a.b is as above
- dot product = scalar product

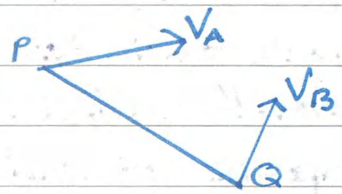
Relative Motion

Displacement of A in terms of B is ${}_A r_B$
Displacement of B in terms of A is ${}_B r_A$

$${}_A r_B = r_A - r_B$$

${}_x r_y$ = x relative to y.

$${}_A V_B = V_A - V_B = {}_A V_E + {}_E V_B$$ where E is a third point



If ${}_A V_B$ is in the direction of PQ then at somepoint A and B will meet

Closest Approach - at time t=0 vectors of A and B

$r_A = (ai + bj + ck)m$ and $v_A = (di + ej + fk)ms^{-1}$
$r_B = (gi + hj + lk)m$ and $v_B = (mi + nj + qk)ms^{-1}$

$r_A(t) = ai + bj + ck + t(di + ej + fk)$
$r_B(t) = gi + hj + lk + t(mi + nj + qk)$

$${}_B r_A = r_B(t) - r_A(t)$$

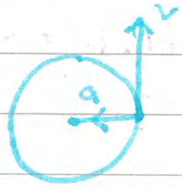
$|\overrightarrow{AB}|^2 = xt^2 - yt + z$ $\frac{d|\overrightarrow{AB}|^2}{dt} = mt + q = 0$ $t = n$

$$\frac{d^2|\overrightarrow{AB}|^2}{dt^2} = x$$

Relationship Between Linear & Angular Speed

$v = \omega r$
$\omega = 2\pi F$
$a = \omega^2 r = v^2/r$

→ v = velocity F = Frequency
r = radius a = acceleration
ω = angular velocity

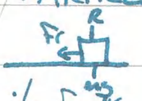
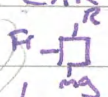
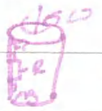
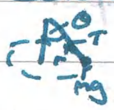


rev min^{-1} = revolutions per minute $\xrightarrow{\times 2\pi}$ rad min^{-1} = radians per minute
↑ ×60 ↓ ÷60

rev s^{-1} = revolutions per second $\xleftarrow{\div 2\pi}$ rad s^{-1} = radians per second

- **Particle on a string** – Force towards the centre is the tension in the string

- **Bead on a wire** – the Force towards the centre is the reaction between bead/wire

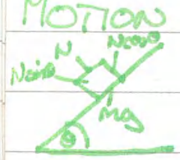
- **Particle on a rotating disc** – Friction is the force between particle/disc; it fails when $F_{max} = \mu R$ towards the centre

- **Car on a circular path** – the centripedal Force is provided by the friction between the tyres and the road if $F \leq \mu R$, it'll slip

- **Particle inside a cylinder** – no slipping: $F \leq \mu R$. about to slip $F = \mu R$. Friction up the cylinder.

- **Conical pendulum** – the centripedal Force is the component of tension

$T\cos\theta = mg$ $T\sin\theta = m\omega^2 r$

Motion on a Banked Curve

$N\sin\theta = mv^2/r$ $\tan\theta = v^2/rg$

$N\cos\theta = mg$ the steeper the slope the faster the corner.

$v = \sqrt{rg\tan\theta}$ TOO FAST = SLIP UP TOO SLOW = SLIP DOWN

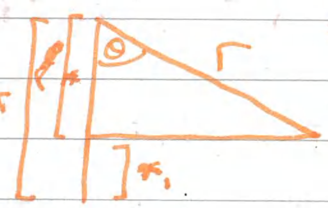


Circular Motion in a Vertical Plane

– we consider motion in which the speed of the particle is not affected by any external forces other than its weight so mechanical energy remains constant

KE + PE = constant

$R - mg\cos\theta = \dfrac{mv^2}{r}$

* $r\cos\theta$
*1 $r - r\cos\theta$

M3/1 RECTILINEAR MOTION

$$a = \frac{dv}{dt} \qquad v = \frac{dx}{dt}$$

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{dv}{dx} \cdot \cancel{d}v = a$$

$$a = v \cdot \frac{dv}{dx}$$

Use $F = ma$

M3/2 SECOND ORDER DIFFERENTIALS

EXPONENTIAL EQUATIONS

$\frac{dP}{dt} \propto P \therefore \frac{dP}{dt} = kP$

$t=0$ $P=N$
$t=1$ $P=M$

$\int \frac{1}{P}\,dP = \int k\,dt$ ...

$\ln P = kt$

$P = e^{kt+c}$

$P = Ae^{kt}$

$N = Ae^{k0}$

$N = A$

$P = Ne^{kt}$

$M = Ne^{k}$

$M/N = e^{k}$

$k = \ln(M/N)$ $[P = Ne^{\ln(M/N)t}]$

HOMOGENEOUS EQUATIONS

$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \qquad a,b,c \in \mathbb{R}$

AUXILIARY = $a\lambda^2 + b\lambda + c = 0$

Roots of Auxiliary	→ General Solution
One Repeated Root, α	$x = e^{\alpha t}(A+Bt)$
Two real roots α, β	$x = Ae^{\alpha t} + Be^{\beta t}$
Two complex roots $p \pm qi$	$x = e^{pt}(A\cos qt + B\sin qt)$
IF $B=0$ $a^2+c=0$ $q=\sqrt{c/a}$	$x = A\cos qt + B\sin qt$

NON-HOMOGENEOUS EQUATIONS $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = At+B$

GENERAL SOLUTION = COMPLIMENTARY FUNCTION + PARTICULAR INTEGRAL

Complimentary Function - solution to corresponding homogeneous eqn.

Particular integral in the form $x = pt + q$

↳ write down trial function, differentiate ×2 sub into original equation and solve.

↳ when roots of auxiliary are zero P.I in form $x = pt^2 + qt$

Sub in values to general solution to get particular solution.

M3/3 SIMPLE HARMONIC MOTION

The motion of a particle is said to be simple harmonic if:
- its acceleration is proportional to distance from a fixed point
- its acceleration is always directed towards the point.
- $d^2x/dt^2 = -\omega^2 x$ - where ω^2 = constant of proportionality

PERIOD [time for one oscillation] $T = \frac{2\pi}{\omega}$

FREQUENCY [1/T] $F = \omega/2\pi$

DISPLACEMENT [from centre] $x = A\sin\omega t + B\cos\omega t$
$= a\sin(\omega t + \alpha)$

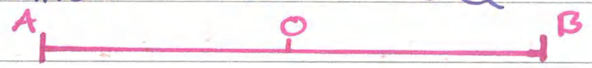
VELOCITY $\dot{x} = v = \delta x/dt$

ACCELERATION $a = \ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 t$

SPEED $v^2 = \omega^2(a^2 - x^2)$

MAX SPEED $v = \omega a$

MAX ACCELERATION $\ddot{x} = -\omega^2 a$

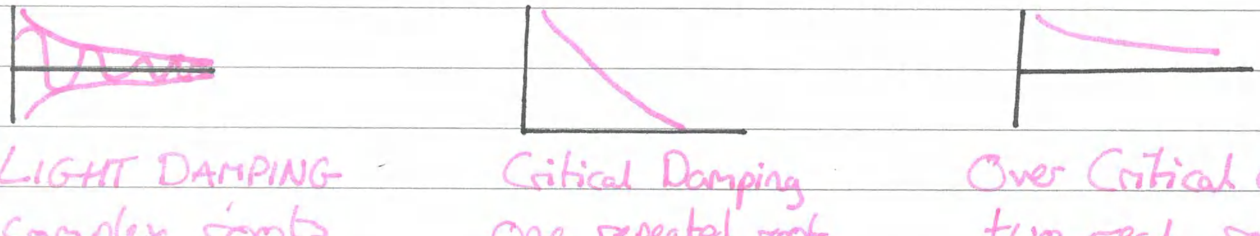


oscillation about O where A and B are max points, OA & OB are ampli
start at extremity - $A\cos\omega t$ start in middle $A\sin\omega t$

Establishing Simple harmonic motion
- use Newtons 2nd Law - $F = ma$
- end up with: $\frac{d^2x}{dt^2} = -\omega^2 x$, $\frac{d^2x}{dt^2} = -\omega^2(x - b)$

Inelastic strings - motion of stretched string is SHM until reaches natural length where it becomes slack and moves as vertical projectile

DAMPING - DEPENDS ON ROOTS OF AUXILIARY EQUATIONS



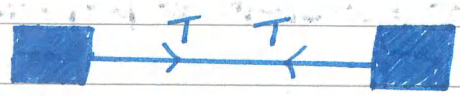
LIGHT DAMPING
complex roots

Critical Damping
one repeated root
no pos value of t
for $x = 0$

Over Critical damping
two real roots

M3/4 Motion of Connected Particles

When a string becomes taught or an impulse is applied, there is a tension induced within it as it resists stretching



The tension creates an impulse which acts on the objects at each end of the string - Impulse tensions.
⇔ NOT TENSIONS - same dimensions as an impulse (Ns)

- Two objects are connected by a string and motion causes the string to be taught, each object experiences a sudden jerk unless the string is elastic.
- Consider the situations just before and after the jerk
- There is a change in momentum due to impulsive tension
- both objects have the same velocity after the string becomes taut.

- Momentum = mv
- Impulse = change in momentum
- Momentum increases in the direction of the impulse
- In direction where no external force acts the momentum is constant
- Particles moving at the end of the string have equal velocity components in the direction of the string

calculate speed before

con of mom applies

M3/5 STATICS

CONDITIONS FOR EQUILIBRIUM

- the sum of the forces is zero
- the sum of moments about any point is zero

MOMENT = Perpendicular Force × DISTANCE

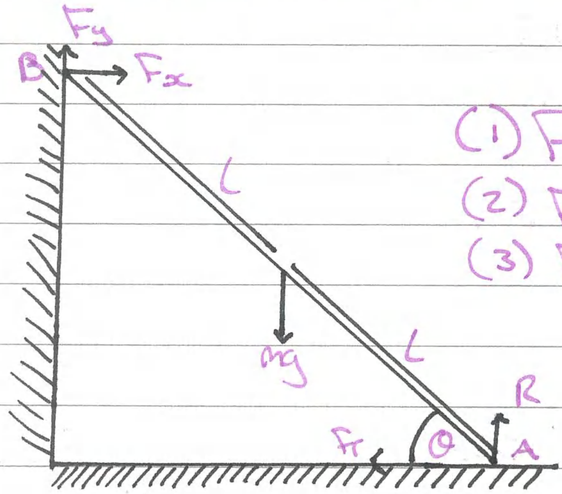
CLOCKWISE MOMENTS = ANTICLOCKWISE MOMENTS

WEIGHT ALWAYS ACTS VERTICALLY DOWNWARDS in the centre of an object.

Reaction at a wall has vertical & horizontal parts

Moments act perpendicular to beam

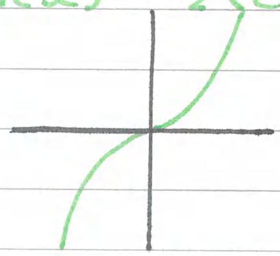
LADDERS - when an object is in limiting equilibrium Frictional force is max : $F_R = \mu R$



(1) FIND MOMENTS ABOUT A
(2) FIND HORIZONTAL FORCES
(3) FIND VERTICAL FORCES

FP3/1 HYPERBOLIC FUNCTIONS

$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$



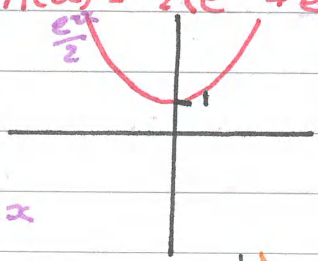
large + values

$\sinh(x) \to \cosh(x) \to \frac{1}{2}e^{x}$

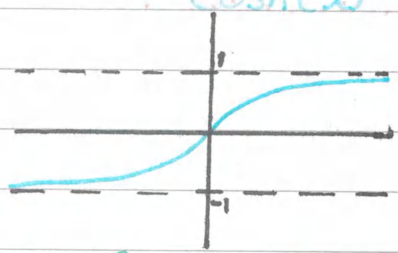
large - values

$\sinh(x) \to -\cosh(x) \to -\frac{1}{2}e^{-x}$

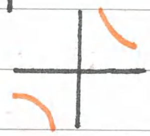
$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$



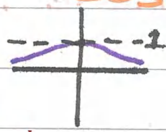
$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{(e^{x} - e^{-x})}{(e^{x} + e^{-x})}$



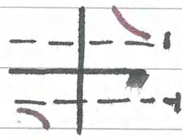
cosech(x)



sech(x)



coth(x)



$\cosh^2(x) - \sinh^2(x) \equiv 1$

$1 - \tanh^2(x) \equiv \text{sech}^2(x)$

$\coth^2(x) - 1 \equiv \text{cosech}^2(x)$

$\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$

$\cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

$\tanh(A \pm B) \equiv \dfrac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$

$\sinh(2A) \equiv 2 \sinh A \cosh A$

$\cosh(2A) \equiv \cosh^2 A + \sinh^2 A$

$\tanh(2A) \equiv \dfrac{2 \tanh A}{1 + \tanh^2 A}$

$F(x)$	$F'(x)$	$\int F(x)\,\delta x$
$\sinh(Kx)$	$K \cosh(Kx)$	$\frac{1}{K} \cosh(Kx) + c$
$\cosh(Kx)$	$K \sinh(Kx)$	$\frac{1}{K} \sinh(Kx) + c$
$\tanh(Kx)$	$K \text{sech}^2(Kx)$	$\frac{1}{K} \ln(\cosh(Kx)) + c$

$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1}) \quad x \geqslant 1$

$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) \quad |x| < 1$

if $p \geqslant 1 \to$

$\cosh x = p$

$\to x = \pm \cosh^{-1} p$

FP3/2 FURTHER INTEGRATION

$\int \frac{1}{ax^2+bx+c}$ $\quad ax^2+bx+c = a((x+p)^2+q)$ use $u=x+p$

$\int \frac{1}{a}\left(\frac{1}{u^2+q}\right)du$

$\int \frac{1}{\sqrt{ax^2+bx+c}}$ $\quad ax^2+bx+c = a((x+p)^2+q)$ use $u=x+p$

$a>0$ $\quad \int \frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{u^2+q}}\right)du$

$\int \frac{1}{\sqrt{ax^2+bx+c}}$ $\quad ax^2+bx+c = -a(q-(x+p)^2)$ use $u=x+p$

$a<0$ $\quad \int \frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{q-u^2}}\right)du$

IF $q>0$ use $u=\sqrt{q}\sinh\theta$

IF $q<0$ use $u=\sqrt{q}\cosh\theta$

IF $a<0$ use $u=\sqrt{q}\sin\theta$

$\int e^{ax}\cos bx\,\delta x$ & $\int e^{ax}\sin bx\,\delta x$, use parts twice

$\int \frac{1}{a+b\cos x}\delta x \quad \int \frac{1}{a+b\sin x}\delta x \quad \int \frac{1}{a+b\cos x+c\sin x}\delta x$

use $t=\tan(\frac{1}{2}x)$ substitution

$I_n = \int_0^1 x^n e^x\,\delta x \qquad I_n = e-n(I_{n-1})$

$I_0 = \int_0^1 e^x\,\delta x \qquad I_0 = e-1$

$y=f(x)$ from $(a,f(a))$ to $(b,f(b))$

$S=\int_a^b \sqrt{1+(f'(x))^2}\,\delta x \qquad A=\int_a^b 2\pi y\sqrt{1+(f'(x))^2}\,\delta x$

$x=f(t)$ from $(f(\alpha),g(\alpha))$ to $(f(\beta),g(\beta))$

$y=g(t)$

$S=\int_\alpha^\beta \sqrt{(f'(t))^2+(g'(t))^2}\,dt \qquad A=\int_\alpha^\beta 2\pi y\sqrt{(f'(t))^2+(g'(t))^2}\,d$

FP3/3 POWER SERIES

TAYLOR SERIES: representation of a function as an infinite sum of terms that are calculated from the values of the functions derivatives at a single point.

MACLAURIN SERIES: A TAYLOR SERIES IS CENTRED AT ZERO.

TAYLORS at $x=a$: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^r}{r!}f^r(a) + \dots$

OR: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \dots + \frac{x^r}{r!}f^r(a) + \dots$

MACLAURINS $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^r}{r!}f^r(0) + \dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^r}{r!} \dots$ For all x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1}\frac{x^r}{r} + \dots$ $|x|<1$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \dots$ For all x

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r\frac{x^{2r}}{(2r)!} + \dots$ For all x

$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r\frac{x^{2r+1}}{2r+1} + \dots$ For all x

$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots$ for all x

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots$ For all x

$\tanh^{-1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2r+1}}{2r+1} + \dots$ For all $|x|<1$

FP3/4 NUMERICAL METHODS

If 'f' is a continuous, $f(\alpha)<0$ $f(\beta)>0$ then $f(x)$ has a root between α and β

If 'α' is a root of: $x=g(x)$ then

$$\text{'}x_{n+1}=g(x_n)\text{'}$$

will converge to α provided $|g'(\alpha)|<1$ and initial value is close to α

NEWTON-RAPHSON METHOD

- named after Isaac Newton & Joseph Raphson
- Is a root finding algorithm that uses the first few terms of the Taylorseries of a function $f(x)$ in the vicinity of a suspected root.

If α is an approximate root of the equation $f(x)=0$ →

$$\text{'}\beta=\alpha-\frac{f(\alpha)}{f'(\alpha)}\text{'}$$

→ is usually a much better approximate root.

The iteration '$x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$' with suitable starting value, usually gives rapid convergence to a root.

FP3/5 POLAR COORDINATES

If point 'P' is represented by (x,y) and $[r,\theta]$

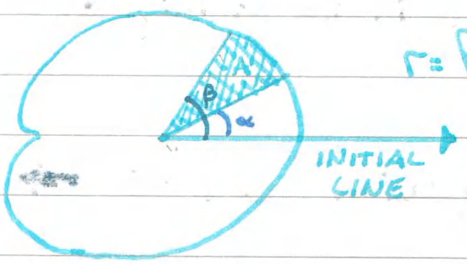
$x = r\cos\theta$ / $y = r\sin\theta$ / $r = \sqrt{x^2+y^2}$ / $\tan\theta = y/x$

$r = F(\theta)$ if $r = 0$ & $\theta = \alpha$ then $\theta = \alpha$ is a tangent at the pole

POINTS OF INTERSECTION $r_1 = F(\theta)$, $r_2 = g(\theta)$ $F(\theta) = g(\theta)$
- but consider whether pole is also point of intersection.

Tangent to polar curve $r = F(\theta)$ PARALLEL to initial line if $\frac{dy}{d\theta} = 0$
where $y = r\sin\theta = F(\theta)\sin\theta$

Tangent to polar curve $r = F(\theta)$ PERPENDICULAR to initial line if $dx/d\theta = 0$ where $x = r\cos\theta = F(\theta)\cos\theta$

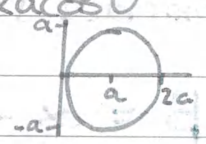


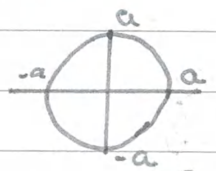
$r = F(\theta)$ $A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

$\alpha = \theta$

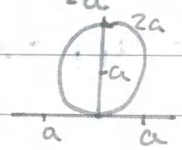
$\beta = \theta$

$r = 2a\cos\theta$ $(x-a)^2 + y^2 = a^2$





$r = a$ $x^2 + y^2 = a^2$



$r = 2a\sin\theta$ $x^2 + (y-a)^2 = a^2$

$$\int \tan^2\theta \, d\theta = \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta$$

$$\int \sin^2\theta \, d\theta = \int \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta) d\theta$$

$$\int \sin^2 2\theta \, d\theta = \int \tfrac{1}{2}(1 - \cos 4\theta)\, d\theta = \tfrac{1}{2}(\theta - \tfrac{1}{4}\sin 4\theta) d\theta$$

$$\int \sin\theta\cos\theta \, d\theta = \frac{1}{n+1}\sin^{n+1}\theta$$