Tutorial Quiz Week 6

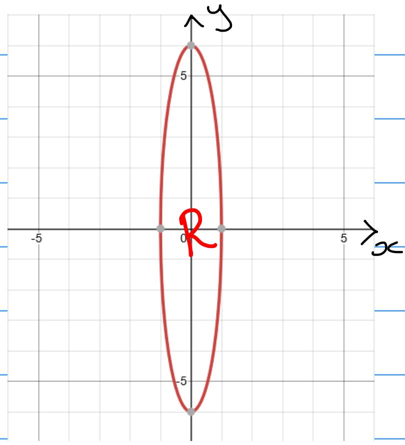
1. Regions when transformation given: For the following Sketch the original region and the new region mapped by the transform, state the new bounding equations.

a) ◦ For a region R bound by the ellipse $x^2 + \frac{y^2}{36} = 1$, and transform $x = u/2$ and $y = 3v$

b) ◦ For the region R bound by the lines $y = -x + 4$, $y = x + 1$ and $y = x/3 - 4/3$, with transformation $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$.

c) ◦ For the trapezoidal region R with vertices given by $(0, 0)$, $(5, 0)$, $(2.5, 2.5)$ and $(2.5, -2.5)$, using the transformation $x = 2u + 3v$ and $y = 2u - 3v$. Solve the integral $\iint x + y\,dA$ using the transformation

a) R is an ellipse, $x = 0 \Rightarrow y = \pm 6$, $y = 0 \Rightarrow x = \pm 1$ hence

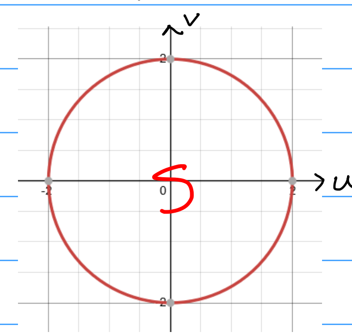


sub transform $x = u/2$ $y = 3v$ into Boundary equation:

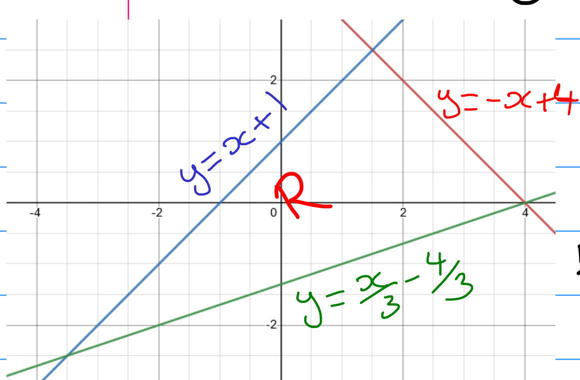
$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$



b) R is a triangle with vertices $(4,0)$, $(1.5, 2.5)$, $(-3.5, -2.5)$



Sub in $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

① $y = -x + 4$

$$\tfrac{1}{2}(u-v) = -\tfrac{1}{2}(u+v) + 4$$

$$\frac{u}{2} - \frac{v}{2} + \frac{u}{2} - \frac{v}{2} = 4$$

$$\Rightarrow u = 4$$

② $y = x + 1$

$$\tfrac{1}{2}(u-v) = \tfrac{1}{2}(u+v)$$

$$\frac{u}{2} - \frac{v}{2} - \frac{u}{2} - \frac{v}{2} = 1$$

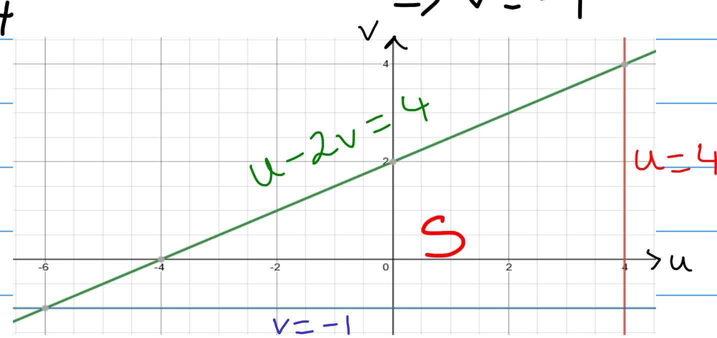
$$\Rightarrow v = -1$$

③ $y = \frac{x}{3} - \frac{4}{3}$

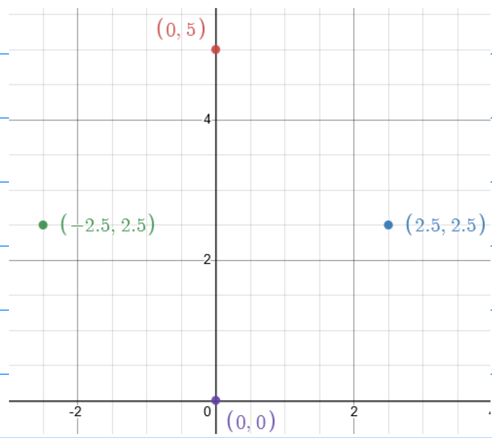
$$\tfrac{1}{2}(u-v) = \tfrac{1}{6}(u+v) - \tfrac{4}{3}$$

$$\frac{u}{2} - \frac{v}{2} - \frac{u}{6} - \frac{v}{6} = -\frac{4}{3}$$

$$\Rightarrow u - 2v = -4$$



c) R trapezoid described by vertices $\Rightarrow$ find governing equations.



line through $(0,0)$ & $(2.5, 2.5)$

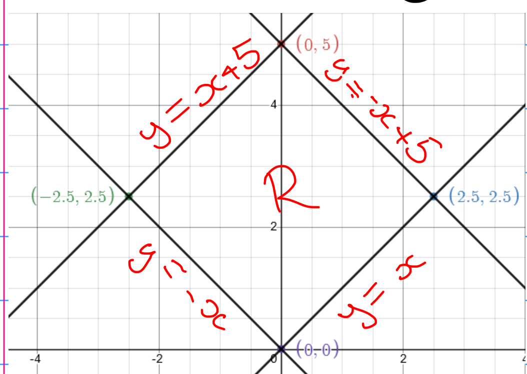
$\Rightarrow \quad y = x$

* notice parallel line, $y = x + 5$

line through $(0,0)$ & $(-2.5, 2.5)$

$\Rightarrow \quad y = -x$

and parallel $y = -x + 5$



Now sub in

$x = 2u + 3v \qquad y = 2u - 3v$

① $y = x$

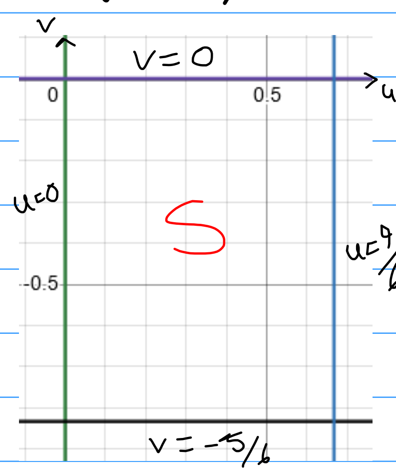
$2u - 3v = 2u + 3v$

$v = 0$

② $y = x + 5$

$2u - 3v = 2u + 3v$

$v = -5/6$



③ $y = -x$

$2u - 3v = -2u - 3v$

$u = 0$

④ $y = -x + 5$

$2u - 3v = -2u - 3v + 5$

$u = 4/5$

N.B. If we then want to integrate over this new region we need

1) Jacobian of transform $\dfrac{\partial(x,y)}{\partial(u,v)}$
2) Limits of integration in u,v plane
3) $f(x,y)$ in terms of (u,v)

2. Transformations when region given: For the following give the transform that maps one region to the other, and draw both regions.

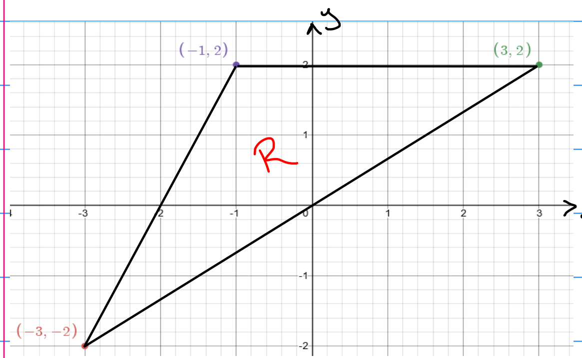
a) ◦ R is the triangle with vertices (3,2), (-1,2), (-3,-2), and S is the triangle with vertices (1,0), (0,1), (-1,0).

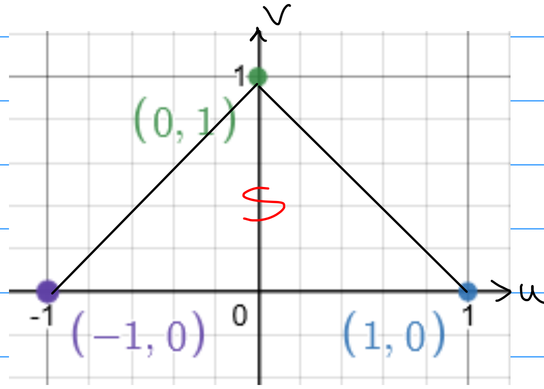
b) ◦ R is the parallelogram with vertices (0,0), (4,2), (3,4) and (-1,2). S is the region defined by $0 \le u \le 10$, $0 \le v \le 5$.

c) ◦ R is the region bound by the equations $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$. S is the region defined by $1 \le u \le 2$, $0 \le v \le \pi$.

d) ◦ R is the unit circle centered at the origin, S is a unit square with vertices (0,0), (0,1) (1,0), (1,1).

a) Drawing both regions first:





Since the map is linear we could use vertices to find the transform, however in general this method won't work $\Rightarrow$ Be careful!

	(x,y)		(u,v)
1	$(-1,2)$	$\longrightarrow$	$(0,1)$
2	$(3,2)$	$\longrightarrow$	$(1,0)$
3	$(-3,-2)$	$\longrightarrow$	$(-1,0)$

Use 1 & 2 to find transform

$$-1 = 0a + 1b$$
$$3 = 1a + 0b$$

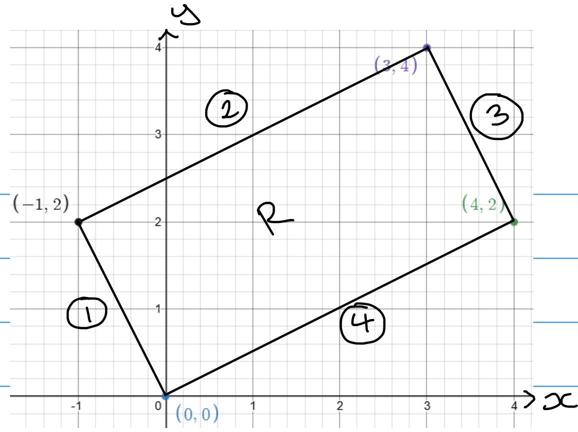
$$2 = 0c + d$$
$$-2 = -c + 0d$$

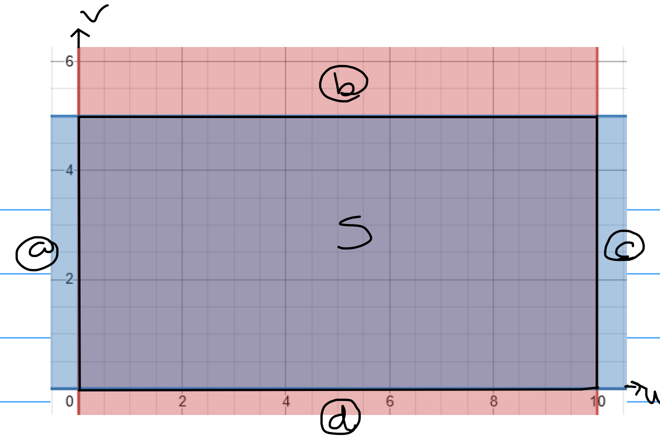
$$\begin{bmatrix} x = 3u - v \\ y = 2u + 2v \end{bmatrix}$$

Check with 3

$\therefore x = 3(-1) - 0 = -3, \quad y = 2(-1) + 2(0) = -2$ ✓

b)





Lets use boundary equations, label 1-4, a-d

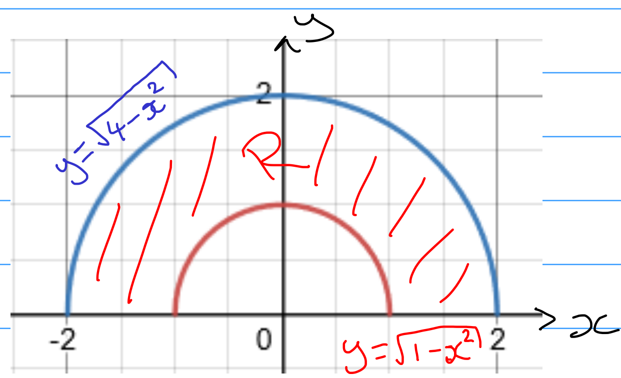
① $y = -2x$	a) $u = 0$
② $y = \frac{x}{2} + \frac{5}{2}$	b) $v = 5$
③ $y = -2x + 10$	c) $u = 10$
④ $y = \frac{x}{2}$	d) $v = 0$

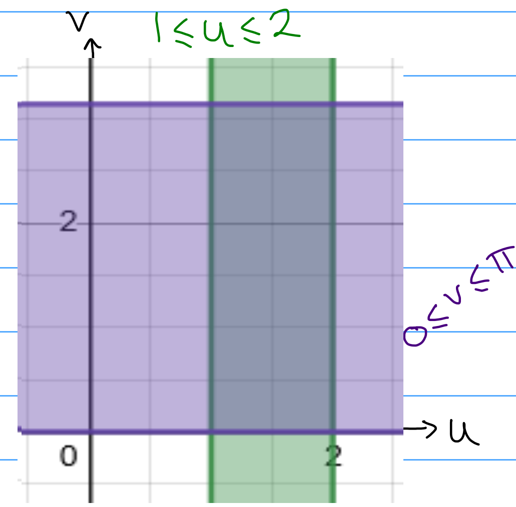
let $u = y + 2x$, $v = 2y - x$, need $x =$, $y =$

so $2u - v = 2y + 4x - 2y + x \Rightarrow x = \frac{1}{5}(2u - v)$

$u + 2v = y + 2x + 4y - 2x \Rightarrow y = \frac{1}{5}(u + 2v)$

c)





Writing out the boundary equation it is easy to see where we should map:

$x^2 + y^2 = 2^2 \longrightarrow u = 2 \qquad u = x^2 + y^2$

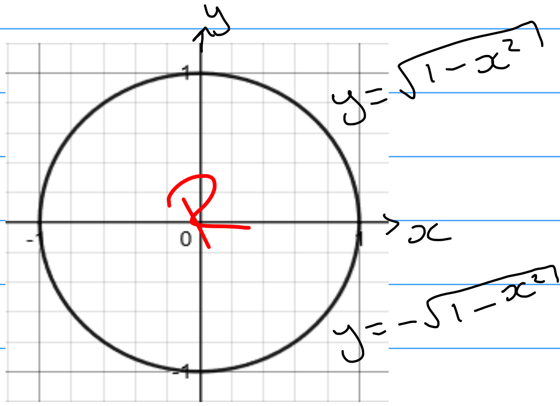
$x^2 + y^2 = 1 \qquad u = 1$

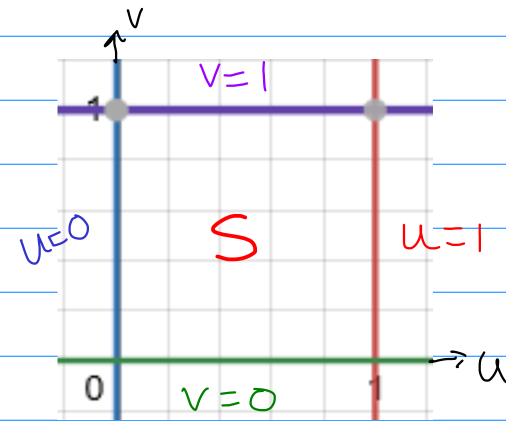
we take $v = 0$ between 0 and π so $v = \tan^{-1}(y/x)$

This is an awkward transform to use, polar would be better!

HARD

d)





Obviously this transform is an interesting choice that we wouldn't use very often. But if we wanted to transform from R to S, what would we have to do?

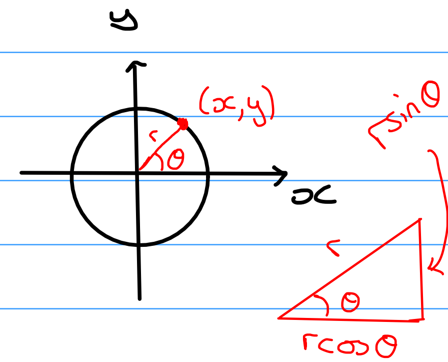
Note: an 'easier' choice may be to map to

$$S=\{(u,v):|u|\leq 1,|v|\leq 1\}.$$

3. Revising Chain rule and Polar Coordinates:

- What are cartesian coordinates in terms of polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$



- What is the chain rule for $\frac{df(g(x))}{dx}$. $= \frac{df}{dg}\frac{dg}{dx}$
- What is the multivariate chain rule for $\frac{df(x(t),y(t))}{dt}$ $= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
- What is the multivariate chain rule for $\frac{\partial f(x(t,s),y(t,s))}{\partial s}$. $= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$
- What is the integral over an area in a general coordinate system? e.g in cartesian $\iint 1dxdy$.

$$I = \iint_A J\,dA$$

- what is the equation of the Jacobian of a transform $(x, y) \to (u, v)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

- What is the equation of the Jacobian of a transform $(u, v) \to (x, y)$

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

- What is the Jacobian of cartesian to polar coordinates? Show workings.

$$J(r,\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

4. Cartesian to Polar Limits: Rewrite the following integral limits into 2d polar coordinates.

a. $\int_0^\infty \int_0^\infty \ldots dydx = \iint \ldots rdrd\theta$

b. $\int_{-\infty}^\infty \int_{-\infty}^\infty \ldots dxdy = \iint \ldots rdrd\theta$

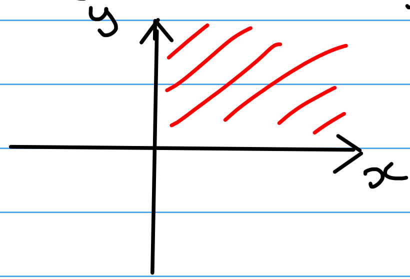
c. $\int_0^1 \int_0^1 \ldots dxdy = \iint \ldots rdrd\theta$

d. $\int_0^1 \int_y^{y^2} \ldots dxdy = \iint \ldots rdrd\theta$

e. $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \ldots dxdy = \iint \ldots rdrd\theta$.

These questions are going to be easier if we look at the regions in cartesian first.

a) $\int_0^\infty \int_0^\infty dxdy$ ← Region bound by $x \geq 0, y \geq 0$
$\Rightarrow$ Positive quadrant

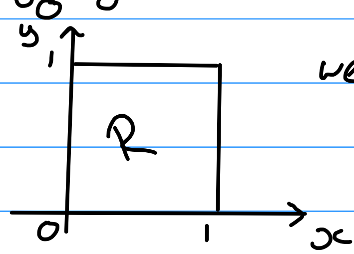


$$\int_0^{\pi/2} \int_0^\infty r\,dr\,d\theta$$

b) $\int_{-\infty}^\infty \int_{-\infty}^\infty dxdy$ ← This is the whole plane! That can be described in polar coordinates uniquely by $\int_0^{2\pi} \int_0^\infty r\,dr\,d\theta$.

If we set $0 \leq \theta \leq \infty$ or $-\infty \leq r \leq \infty$ we'd end up overlapping the plane multiple times.

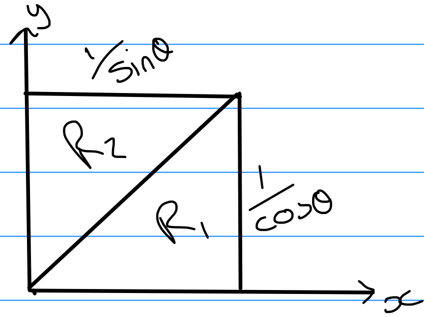
c) $\int_0^1 \int_0^1 dxdy$ ← unit square in the positive quadrant



we need to describe this in polar coords. Plugging $x = r\cos\theta, y = r\sin\theta$ into upper limits gives

$r\cos\theta = 1 \quad r = \frac{1}{\cos\theta} \qquad r\sin\theta = 1 \quad r = \frac{1}{\sin\theta}$

These upper limits now need us to split R into 2 regions

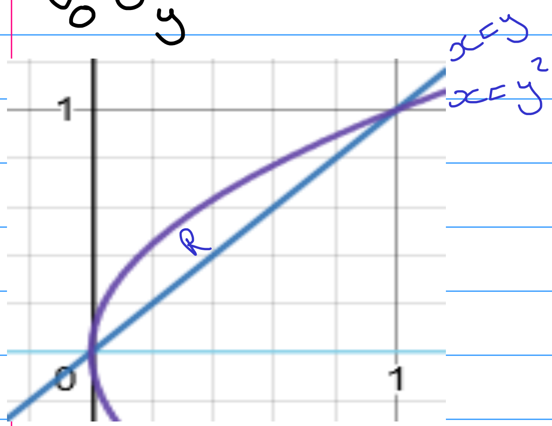


R_1 defined with $0 \le \theta \le \pi/4$

R_2 with $\pi/4 \le \theta \le \pi/2$

Hence $\displaystyle\int_0^{\pi/4}\int_0^{\frac{1}{\cos\theta}} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} r\,dr\,d\theta$

d) $\displaystyle\int_0^1\int_y^{y^2} dx\,dy$



It's easy now to see $\pi/4 \le \theta \le \pi/2$, for radial values our lower limit is 0, upper limit given by $x = y^2$, sub in values:

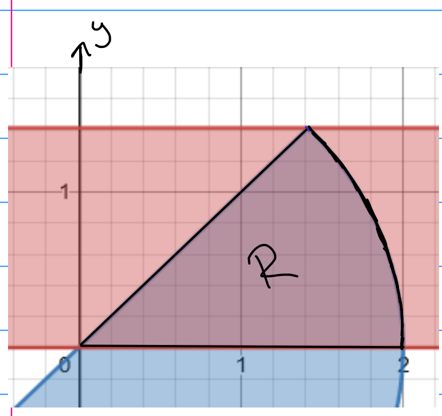
$r\cos\theta = r^2\sin^2\theta$

$r = (\sin\theta\tan\theta)^{-1}$

$\displaystyle\int_{\pi/2}^{\pi/4}\int_0^{(\sin\theta\tan\theta)^{-1}} r\,dr\,d\theta$

e) $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dx\,dy$. The region here is an 8^{th} of the circle with radius 2,

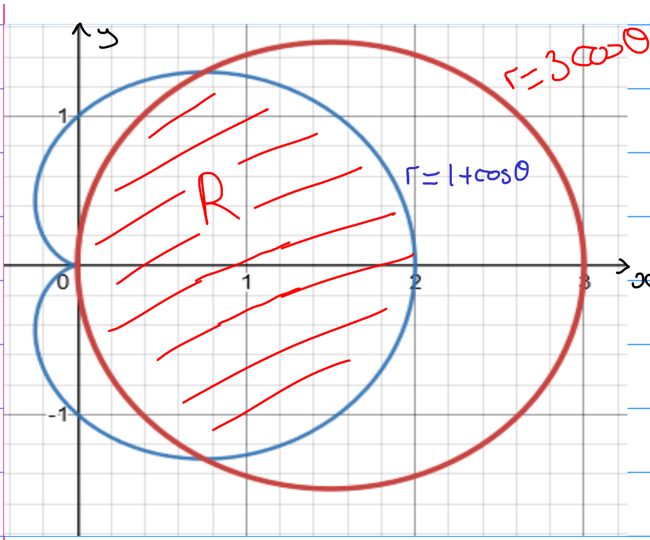
It is easy to see $0 \le \theta \le \pi/4$ and $0 \le r \le 2$.



$\displaystyle\int_0^{\pi/4}\int_0^2 r\,dr\,d\theta$

5. Polar integration: Evaluate the area enclosed by the circle $r = 3cos(\theta)$ and $r = 1 + cos(\theta)$.

- Sketch the regions
- Calculate the points of intersection
- What symmetries or separations of the region can you use to simplify the integral
- Write down the limits of integration $\iint r dr d\theta + \iint r dr d\theta$.
- Evaluate the integral.

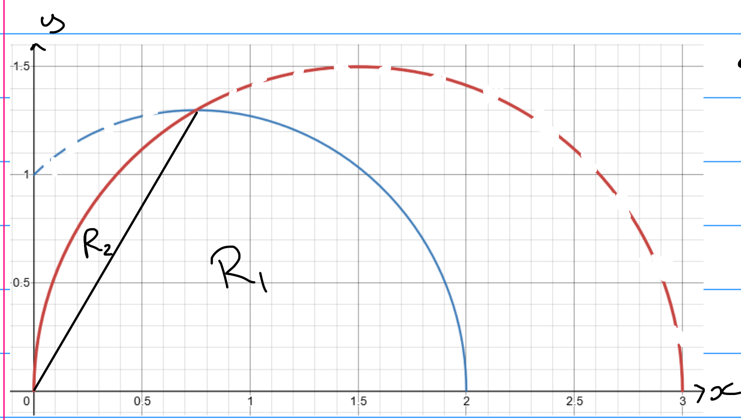


The region bound by the two curves is shaded in red. To find points of intersection set eqtns equal:

$$3\cos\theta = 1+\cos\theta \qquad r = 1+\cos(\pi/3)$$
$$\cos\theta = 1/2 \qquad r = 1.5$$
$$\theta = \pm \pi/3$$

Visually we can see the shape is symmetric about x axis, so we can compute the integral in the +ve quadrant.

We also want to split the region by the upper limits.



Since the intersection is at $\theta = \pi/3$;

$$R_1: 0 \le \theta \le \pi/3$$
$$R_2: \pi/3 \le \theta \le \pi/2$$

Both regions have lower limit $r=0$, and upper limits are $r = 1+\cos\theta$, $r = 3\cos\theta$ respectively. Hence

$$A = 2\left(\int_0^{\pi/3}\int_0^{1+\cos\theta} 1\, r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{3\cos\theta} r\,dr\,d\theta\right)$$

$$= 2\left(\int_0^{\pi/3} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \tfrac{9}{2}\cos^2\theta\,d\theta\right)$$

use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 2\left(\pi/4 + \tfrac{9}{16}\sqrt{3} + \tfrac{3}{8}\pi - \tfrac{9}{16}\sqrt{3}\right) = 5\pi/4 \text{ square units}$$